Thale's theorem, 2D stress-states, and the Mohr's circle

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Abstract

Abstract: We want to emphasize the importance of considering Thale's Theorem with a mathematical representation of a two-dimensional physical-science concept: The Mohr's circle, which is used in stress analysis in two dimensions. The stress analysis is described in detail for the application case in *Engineering Geology*.

Keywords: Thale's theorem, Mohr's circle, stress state, engineering geology

1 Thale's theorem (TT)

In geometry, Thale's theorem, (which should not be confused with the *basic proportionality theorem*) which is about *inscribed angles in a semicircle*, states that if A, B, and C are distinct points on the perimeter of a circle $\mathcal{C}$, where the segment $\overline{AB}$ is the diameter and $\overline{CB}$ is a secant of that circle, the angle $\angle ACB$ is a right angle. Figure 1 shows the graphical interpretation of the above statement for the *unit circle* ($\mathcal{C}_1$) whose centre is at the origin of a Cartesian coordinate system.

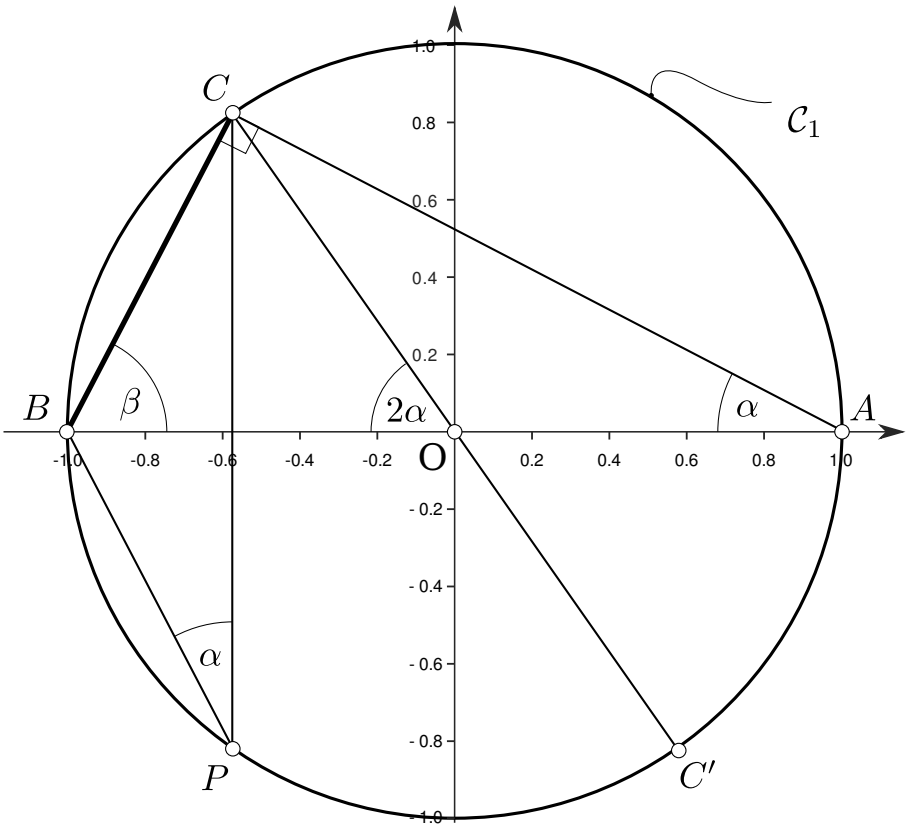

Figure 1: Thale's theorem in $\mathcal{C}_1$.

Point C can be located at any place in the perimeter of the circle, and at any of its positions,

$$\angle CAB = \frac{1}{2}\angle COB.$$

An extension of this theorem says that given three points C, P, and B on any circle $\mathcal{C}$ with a centre at O, the angle $\angle COB$ is twice as large as the angle $\angle CPB$ (also shown in Figure 1), which is derived from the *inscribed angle* theorem.

The inscribed angle theorem relates the measure of an inscribed angle to that of the central angle subtending the same arc. Here, the common chord at the circle is segment $\overline{CB}$. Chords $\overline{PB}$ and $\overline{PC}$ share an endpoint P, and arc $\overline{CB}$ is subtended by two angles, where one angle is an *inscribed angle* $\angle CPB$, and the other angle is the *central angle* $\angle COB$.

This point P can be located at any position within the perimeter of the circle, and in any of its positions,

$$\angle CPB = \frac{1}{2}\angle COB.$$

The theorems apply to any circle ($\mathcal{C}$), not just the unit circle ($\mathcal{C}_1$).

2 Stress state (SS)

In the context of engineering geology, a **stress state**, named S, is a material mechanical condition that represents the forces applied to an infinitesimal

volume (a point in $\mathbb{R}^3$). Bidimensional reduction is widely used by practitioners, as they can simplify a real three-dimensional stress state into a two-dimensional state by making use of stress symmetries, such as planar or axial symmetries. Additionally, the geometrical symmetries of the solid under stresses are important to attain that stress symmetry. Therefore, in two dimensions, a stress state is that material condition that measures the forces applied to an infinitesimal area (a point in $\mathbb{R}^2$).

This reduction into a two-dimensional space is possible in special solid mechanical pieces (*i.e.*, in mechanical engineering) that sustain forces applied axially or transversely, normal or parallel to a geometrical symmetric solid.

In civil and geological engineering, *i.e.*, in geotechnical engineering (soil and rock mechanics), a two-dimensional stress state can be assumed in certain circumstances, *e.g.* cutting the terrain vertically in a vertical plane or looking from above the terrain, into the terrain surface and below it in an horizontal plane.

3 The Mohr's circle (MC)

The two-dimensional version of **Mohr's circle** is a graphical representation of a two-dimensional stress state in the *trace of a plane*, where the trace is the line in which the stress plane intersects with the other reference plane. It is a particularly attractive representation form for any stress state in a *plane deformation state* or *plane stress state*. The Mohr's circle is not simply

a plot; it is considered a graphical calculator, and its applications can be extended to any two-dimensional second-order tensor.

The Mohr's circle is drawn in a *stress-state space* defined by the Cartesian product of two variables $(\sigma_{\mathrm{n}}, \tau)$; the horizontal-axis (of a Cartesian orthogonal bi-dimensional coordinate system) represents the normal stress applied to any plane (σ_{n}); the vertical-axis, represents the tangential stress, also called *shear stress*, that is parallel to that plane (τ).

To represent a stress state, it is not sufficient to have the pair $(\sigma_{\mathrm{n}}, \tau)$; it is also necessary to relate the orientation of the plane where those two stresses act. The angle α is the angle between the direction of the *major principal stress* and the *plane trace* (the line that represents the plane in two dimensions). The direction parallel to the trace of the plane where the shear and normal stresses act is always parallel to the `x-axis` in this representation; here, the σ_{n} axis and the angles are measured counterclockwise from the x-axis to the stress direction in question.

The Mohr's circle centre is always touching the horizontal-axis due to the *shear stress symmetry*, which is also reflected in the symmetry of the matrix representation of a two-dimensional second-order tensor.

For the sake of the importance of the *principal stresses* and because the centre of the circle is always on the `x`–axis, the circle will be defined in this context by the two points that form the diameter of the circle at the `x`–axis. We call these two points $B : (\sigma_3, 0)$ and $A : (\sigma_1, 0)$, where σ_1 and σ_3 are called the principal-major and principal-minor stresses, respectively, as

shown in Figure 2.

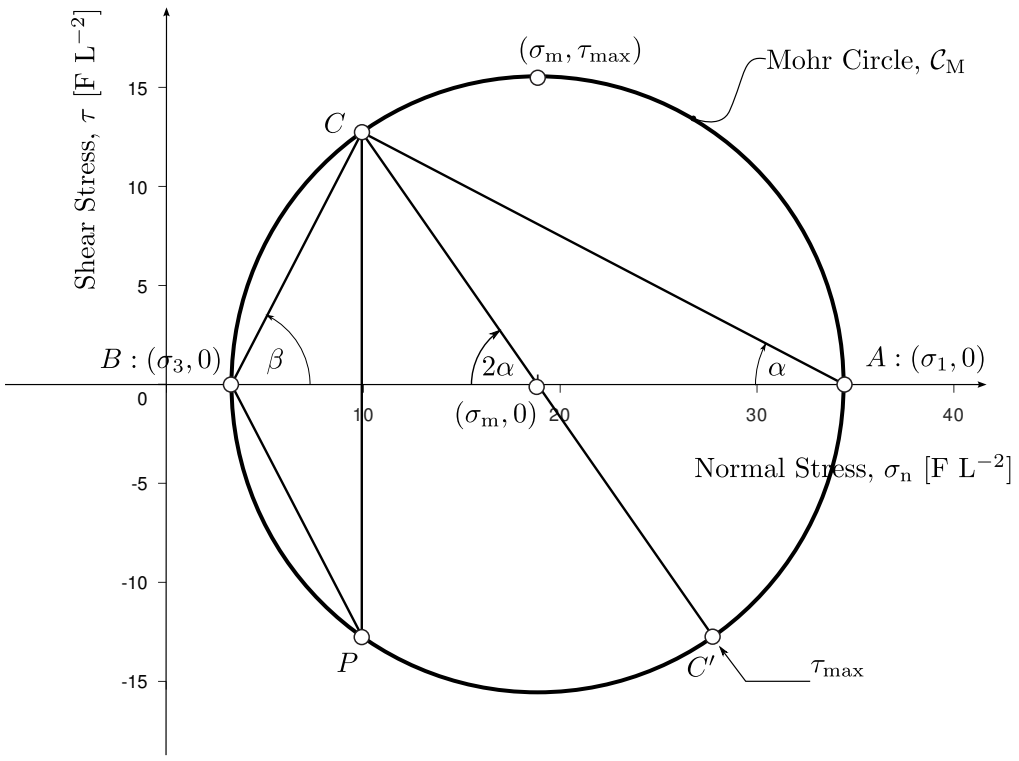

Figure 2: Two-dimensional stress state represented by the Mohr's circle.

If C is a point given by the pair of stress magnitudes $(\sigma_{\rm nC}, \tau_{\rm C})$ in the stress-state space and belonging to a circle $\mathcal{C}$, then we will need the line starting from C to touch the `x`-axis to completely define the stress state.

The inclination of that line will determine the inclination of the shear plane relative to the direction of the major or minor principal stresses, and we call the inscribed angles α and β.

Here, the statement of the Thale's theorem extension where the inscribed

angles are half the central angles is realized (Figure 2).

If one knows the complete stress state from its two principal stresses, which are the major stress and the minor stress, *i.e.*, σ_1 and σ_3 then, the Mohr's circle ($\mathcal{C}_{\mathrm{M}}$) is the object such that it can be defined as follows:

$$\mathcal{C}_{\mathrm{M}} = \{\sigma_3, \sigma_1\}.$$

Because the stress state is a circle, one can represent the stress state as many combinations of variables that are appropriate to define a circle (whose centre passes through the horizontal axis); some of those forms are the following.

1. Mohr's circle defined by the abscissas of the leftmost and rightmost points on the x-axis, i.e., $x_{1,0}$ and $x_{2,0}$, respectively.

$$\mathcal{C}_{\mathrm{M}} = \{x_{1,0} \leftarrow \sigma_3 \text{ and } x_{2,0} \leftarrow \sigma_1\}.$$

2. Mohr's circle defined by its centre point abscissa on the x-axis (x_o) and its radius (r).

$$\mathcal{C}_{\mathrm{M}} = \{x_o \leftarrow \sigma_{\mathrm{m}} \text{ and } r \leftarrow \tau_{\max}\};$$

where

$$\sigma_{\mathrm{m}} = \frac{1}{2}(\sigma_1 + \sigma_3),$$

$$\tau_{\max} = \frac{1}{2}(\sigma_3 - \sigma_1)$$

which are the *mean normal stress* and *maximum shear stress*, respectively.

3. Mohr's circle defined by a point that belongs to the circle and the slope of the radius from that point, *e.g.*, the point C and the slope of $\overline{OC}$.

4. Mohr's circle defined by the location of its diameter segment (two points that form the diameter of the circle), *e.g.*, C and C'.

5. Mohr's circle defined by the location of two points at the perimeter of the Mohr's circle that does not form the diameter of the circle, *e.g.*, C and P.

Each case of defining C_{M} will be used according to the variables that are known.

Additionally, the infinite planes, represented by the trace lines, at which the normal and shear stresses can define the same stress state, are equivalently represented by the infinite points that the perimeter of the Mohr's circle can sustain. Therefore, a point in the perimeter of the Mohr's circle represents one inclined plane (one trace line), i.e., only one means in which to visualize a stress state through the magnitudes σ_{n} and τ.

Since geometry rules are universal, Thale's theorem and its extension, which is presented in the first section of this text, also apply to the stress-state space and to the Mohr's circle.

4 TT, SS, and MC

We said that any point on the Mohr's circle perimeter represents a different point-of-view of the same stress state. Here is where the triangle of Thale's theorem is constructed. Referring to Figure 1, point C of the triangle is any point on the Mohr's circle perimeter, and points A and B are on the `x`-axis defining the circle diameter.

Using Thale's theorem, angles α and β are both complementary angles of the other; then, lines $\overline{AC}$ and $\overline{BC}$ are perpendicular. In terms of the Mohr's circle, lines $\overline{AC}$ and $\overline{BC}$ are the inclinations of the plane trace lines of two perpendicular planes where the stresses act.

5 Example

This is an example of the stress state of rock in a shear geological failure zone of the Earth.

5.1 Statement

The trace of a vertical shear failure on a flat terrain has an azimuth direction of N30°E (from the geographical north, 30 degrees to the east). The geologists determined that in that plane, the blocks are displaced in the left-hand sense (Figure 3), which means that the western block displaces to the southwest relative to the eastern block. Geological observations in the field deduced

that the tangential stress magnitude (τ_{E}) is 13 MPa and the normal-to-plane stress magnitude (σ_{nE}) is 10 MPa in these senses, as shown in the same figure.

Considering a Navier–Coulomb failure criterion with a friction angle of 35 degrees, what is the stress state that caused the rock-material rupture along that shear plane and that sense of displacement?

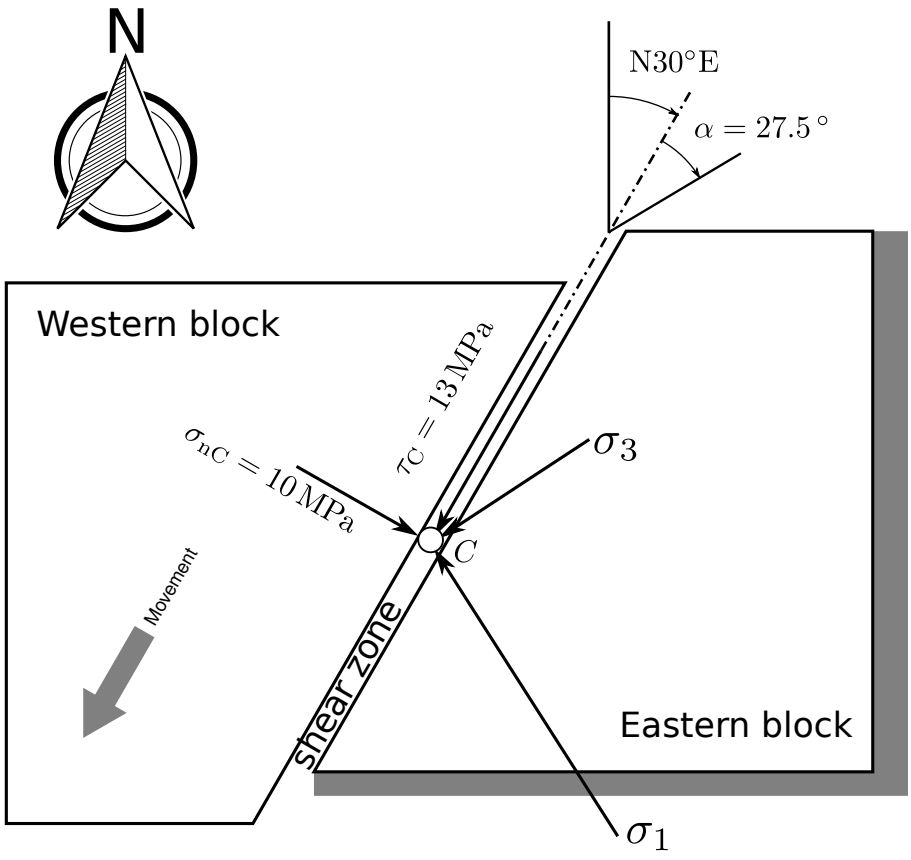

Figure 3: Plant view of the state-stress in a geological vertical shearing failure-zone.

Solution

Assuming that the principal stresses are horizontal and that the failure plane is vertical, the problem can be solved in two dimensions, looking from the air to the ground.

To determine the *stress state*, the two principal stresses magnitudes and the orientations of those two principal stresses need to be determined.

Here, we do not know the magnitudes of either of the principal stresses (major σ_1 and minor σ_3) that cause the failure. Additionally, we do not know its orientations.

Because we know the magnitude of the normal and shear stresses at point C, *i.e.*, $(\sigma_{\mathrm{nC}}, \tau_C)$, we know a point in the Mohr's circle which in turn is the pair of stresses that causes the rock-rupture (the failure). Additionally, we know the orientation of the shear plane where both stresses are acting. Therefore, this is the *third case* among the five listed at the end of Section 3 within this document, where the Mohr's circle is defined by a point that belongs to the circle and the slope of the radius from that point.

Figure 4 presents the two-dimensional stress-state space, where the x-axis is the normal stress and the y-axis is the shear stress. In engineering geology, the positive values of the x-axis represent compressive stresses. Because σ_{nC} is compressive in this example, as shown in Figure 3, the sign of this magnitude is positive. For the shear stress, we do not have direct information about the sign; thus, we can initially define the following two points: one C with coordinates $(10, +13)$ and the other P with coordinates $(10, -13)$

according to the magnitudes defined above. Only one solution is possible, but currently, we have two points that can produce two solutions. In Figure 4, the two points are depicted.

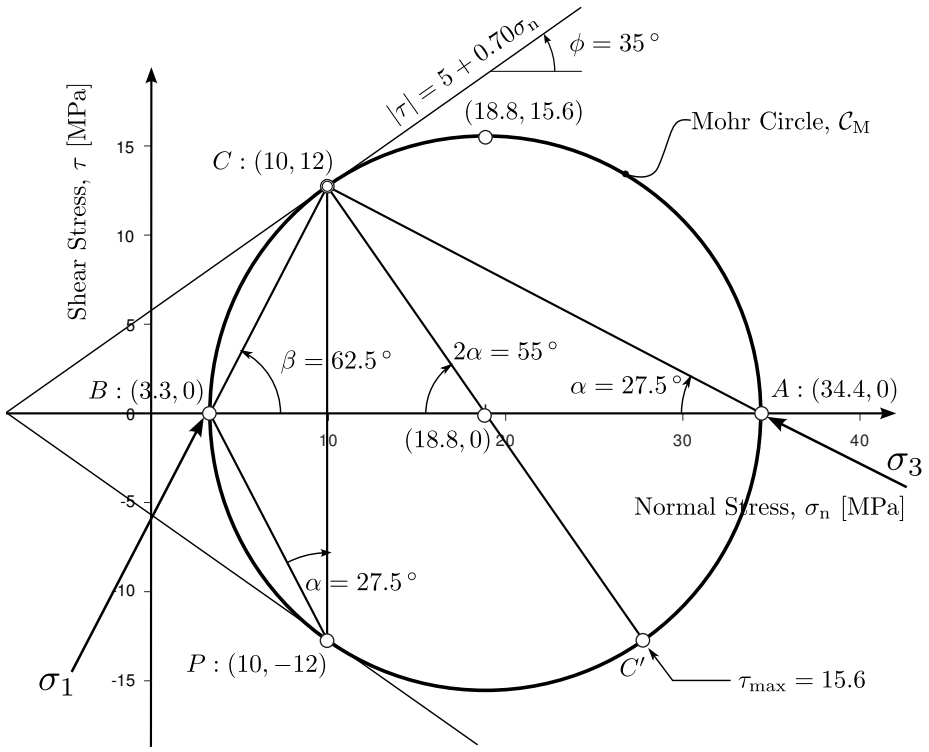

Figure 4: Stress state of the case shown in Figure 3.

At a point, such as C, infinite circles can pass, but because one Mohr's circle condition is that its centre is at the x-axis, our infinite set is reduced. However, there are still infinite circles possible under that condition. To reduce the number of possible circles even more, we make use of the failure criterion. In this case, the problem statement recommends using the Navier–Coulomb criterion from among the many that exist in engineering geology.

In the two-dimensional stress-state space, this failure criterion is represented by an inclined line that touches the failure point $(\sigma_{\mathrm{nC}}, \tau_C)$ tangentially (a mechanical concept). The slope of that line is equal to $\tan\phi$, where ϕ is called the *material friction angle*, which is a known value here due to the Navier–Coulomb criterion. This failure criterion states that the absolute value of the shear at failure (*i.e.*, shear strength) increases linearly as the signed normal stress increases.

Now, we make use of Thale's theorem. Referring to Figure 1, the failure point $(\sigma_{\mathrm{nC}}, \tau_C)$ is point C. To accomplish C as the tangent point of the line that represents the Navier–Coulomb criterion line,

$$2\alpha + \phi = 90\,^\circ$$

should be realized. Because ϕ is known, this angle α is calculated.

The two rays of the known α angle, here equal to 27.5 °, from point C, forming 90 degrees between them, emanate downwards until they intersect the x-axis at points A and B. Similarly, from P, two rays emanate upwards until they intersect the x-axis at points A and B. With this angle, the circle is defined, and the stress state is established.

However, although the stress state is defined, we have two possible situations for the same stress state with respect to the sign of the shear stress, *i.e.*, point C and point P.

To solve this, we should consider the sense of the displacement. If point

C is the correct solution, then the western block displaces to the southwest; if point P is the correct solution, then the western block displaces to the northeast. Since the field observations of this example state that the western block displaces to the southwest, the correct solution is the point C with positive signed shear stresses in the matrix representation of the second-order tensor representation.

The angle α is the acute angle between the line $\overline{CA}$ and the x-axis. We are required to inform the orientation of the major principal stress with a clockwise angle from the geographical north, *i.e.*, its azimuth. Because we know the azimuth of the trace of the shear plane, i.e., N30 °E, we add 27.5 °in a clockwise sense to obtain N57.5°E.

Similarly, and keeping in mind our Thale's theorem solution depicted in Figure 1, the orientation of the minor principal stress can be computed as the azimuth by calculating the angle β, which is the complementary angle of α, 62.5 °here. Then, the azimuth of this stress in the sum of the azimuth of the trace of the shear plane plus β counterclockwise to obtain N30°E - 62.5°= N32.5°W.

The principal stress magnitudes can be inferred by solving the intersections of the circle with the x-axis; thus, the minor stress magnitude is $\sigma_3 = 3.3\,\mathrm{MPa}$, and the major stress magnitude is $\sigma_1 = 34.4\,\mathrm{MPa}$, which are the magnitudes we were aiming to obtain.

The solution can be written as follows: the stress state that caused the blocks to be left-sense displaced along the shear vertical plane, under the

Navier-Coulomb failure criterion, is a major principal compressive stress of 34 MPa oriented N57.5°E and a minor principal compressive stress of 3 MPa oriented N32.5°W.

6 Epilogue

The role that Thale's theorem takes in the Mohr's circle concept can be just a small part of the whole theory the circle represents, but Thale's theorem can help greatly in the memorization of the procedure involved in the graphic tool when dealing with the Mohr's circle.

Memorizing the postulates of Thale's theorem is recommended for applying Mohr Circle; *e.g.*, then the stress states are solved in two dimensions, which may be easy to deal with without ambiguities or errors.

Postgraduate students and engineers (*e.g.*, engineering geologists, as the example here) coming from mechanical disciplines often confuse the meaning of the angle α and its relation with angle β because they do not remember Thale's theorem.

Another important error commonly present by those users is that they violate Thale's theorem when they sketch the Mohr's circle without a ruler and compass during field-work calculations. They do not recognize the importance of obtaining a right triangle inside the circle, even though the values of the normal and tangent stresses are commonly approximately guessed.

This example can motivate students to learn with Thale's theorem. Even

though Thale's theorem may be easy to assimilate, we consider that the mechanical and the geological concepts explained here may be difficult to understand; thus, the bibliographic reference described in 6 may be useful.

Recommended readings

[1] G. Mandl. *Rock joints: the mechanical genesis*. Springer, Berlin, 1 edition, 2005.

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[3] R.H.G. Parry. *Mohr circles, stress paths and geotechnics*. Spon Press, London, 2 edition, 2004. `https://www.routledge.com/Mohr-Circles-Stress-Paths-and-Geotechnics/Parry/p/book/9780367871253`.